Improved Provable Reduction of NTRU and Hypercubic Lattices

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Main results

Motivating question: can we provably show that some lattices can be reduced using SVP oracles in dimensions substantially smaller than their rank n?

Previous work:

- Heuristic estimates.
- Dimension $n/2$ SVP oracles are enough to reduce $\mathbb{Z}^n$ [Duc23].

Our results:

- Oracles in [Duc23] can be relaxed to approximate-SVP oracles.
- For many NTRU instances: $n/2$ is also sufficient.

⚠ We **do not** claim any security loss on $\mathbb{Z}$LIP or NTRU based schemes. ⚠

Roadmap

- I. Intro: Building Blocks
- II. A Primal/Dual Reduction Framework
- III. Application: Hypercubic Lattices
- IV. Application: NTRU Lattices
- V. Comparison with Heuristic Reduction

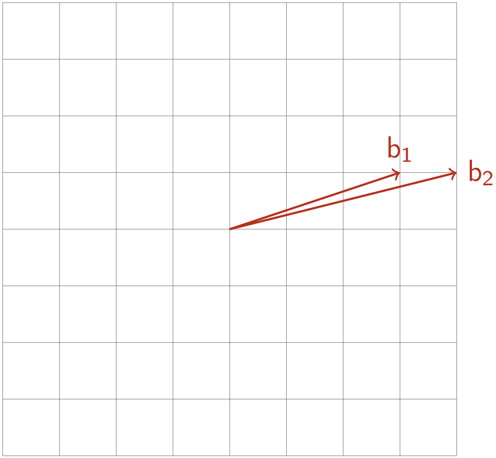
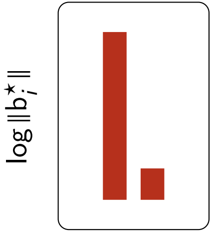
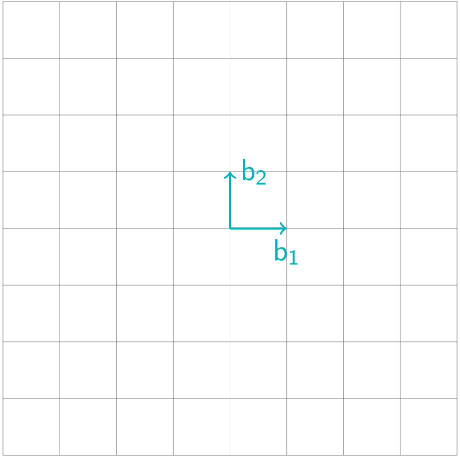


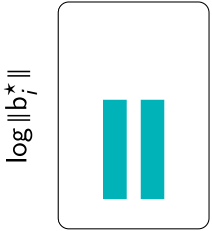
Figure: Gram-Schmidt profile



Convert a bad basis B into...

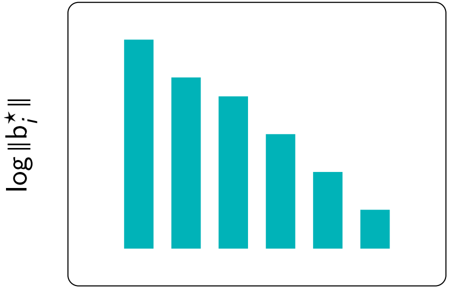
Lattice algorithms

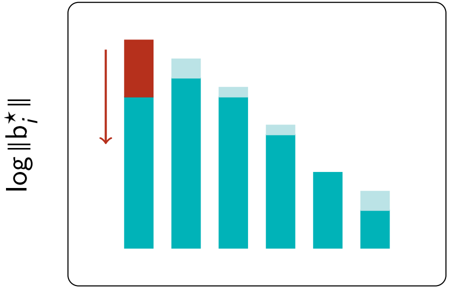




… a better basis B.

Building block: SVP Reduction





γ-SVP oracle

Outputs a basis B whose first Gram-Schmidt norm is $\|b_1^\star\| \leq \gamma \lambda_1(\mathscr{L}(B))$.

Two very special lattices

Hypercubic Lattices:

- Orthonormal basis
- Used in *Lattice Isomorphism Problem* ($\mathbb{Z}$LIP) and HAWK [DvW22, DPPvW22]

NTRU Lattices:

- Module structure
- Used in many schemes and standards: NTRU, Falcon, ... [HPS98, CDH+20, FHK+19]

- In general, lattice reduction estimates are heuristic and rely on low-dim experiments and predictions on the behaviour of lattice algorithms (BKZ).

Provable reduction with smaller blocks: what do we know?

Question

Is it possible to provably solve SVP in special families of lattices of rank n using only SVP-oracles in dimension $\beta = \alpha n$ for a constant $\alpha < 1$?

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Is it possible to provably solve SVP in special families of lattices of rank n using only SVP-oracles in dimension $\beta = \alpha n$ for a constant $\alpha < 1$?

For Hypercubic Lattices:

- In 2023, Ducas proved that $\alpha = \frac{1}{2}$ suffices [Duc23].

For NTRU Lattices:

- Until now, no α better than 1.
- In 2006, Gama, Howgrave-Graham and Nguyen conjectured $\alpha < 1$ [GHN06].

Duality

Dual lattice

Every lattice Λ can be paired up with a dual lattice $\Lambda^\times$.

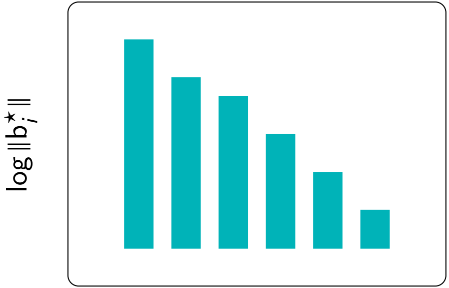
Dual basis

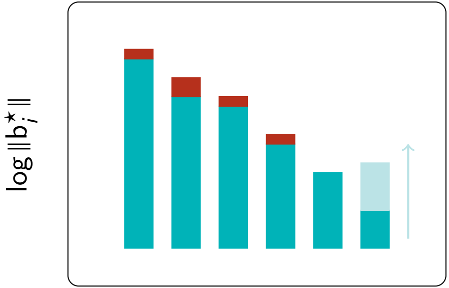
Every lattice basis $(\mathrm{b}_1, \ldots, \mathrm{b}_n)$ can be paired up with a dual basis $(\mathrm{d}_1, \ldots, \mathrm{d}_n)$, which is such that

$$\|\mathrm{b}_n^\star\|^{-1} = \|\mathrm{d}_1^\star\|.$$

Hypercubic lattices are isodual! $(\Lambda = \Lambda^\times)$

Building block: Dual-SVP Reduction





γ-Dual-SVP oracle

Outputs a basis B whose first dual Gram-Schmidt norm is

$$\|d_1^\star\| = \|b_n^\star\|^{-1} \le \gamma \lambda_1(\mathscr{L}(B)).$$

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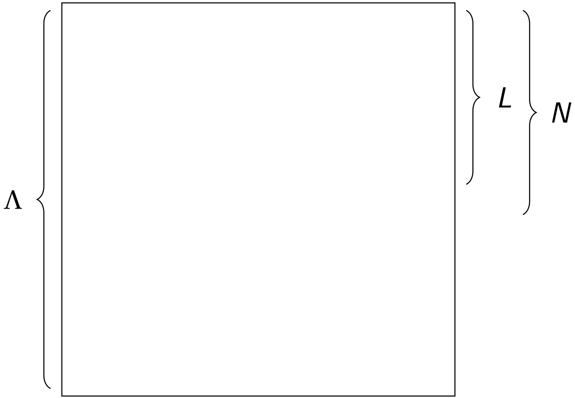
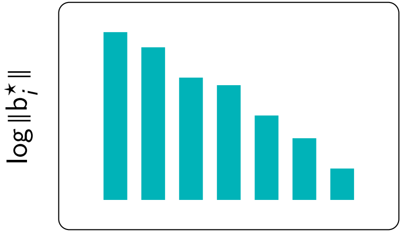
Primal/Dual Reduction: A nice tool for provable reduction

$$\Lambda = \mathscr{L}(\mathbf{b}_1, \dots, \mathbf{b}_n) \qquad L = \mathscr{L}(\mathbf{b}_1, \dots, \mathbf{b}_k) \qquad N = \mathscr{L}(\mathbf{b}_1, \dots, \mathbf{b}_{k+1})$$



Dimension $n = 2k + 1$

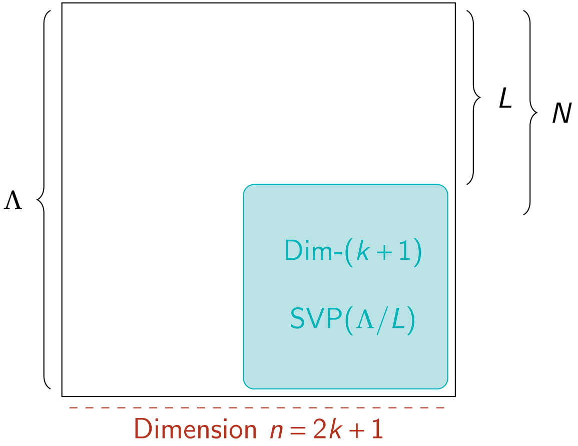


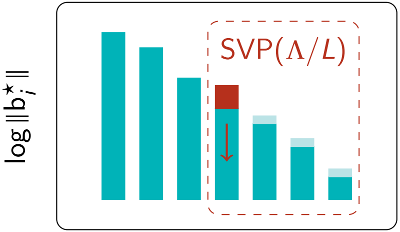
We know that

$$\mathrm{vol}(N) = \mathrm{vol}(L)\|\mathbf{b}_{k+1}^\star\|.$$

Slide-inspired Reduction: Primal step

$$\Lambda = \mathscr{L}(\mathbf{b}_1, \ldots, \mathbf{b}_n) \qquad L = \mathscr{L}(\mathbf{b}_1, \ldots, \mathbf{b}_k) \qquad N = \mathscr{L}(\mathbf{b}_1, \ldots, \mathbf{b}_{k+1})$$



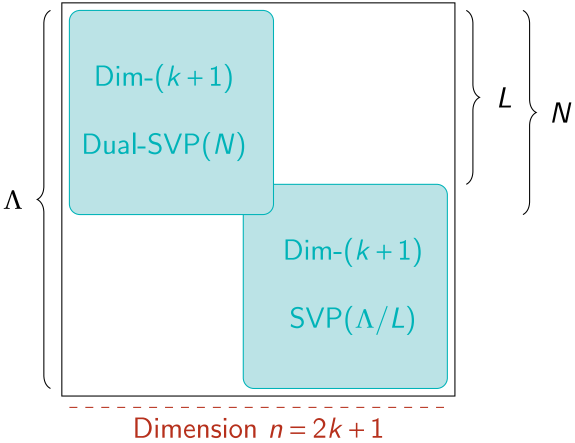


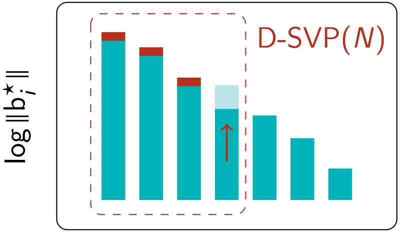
After SVP-reduction:

$$\|\mathbf{b}_{k+1}^\star\| = \lambda_1(\Lambda/L).$$

Slide-inspired Reduction: Dual step

$$\Lambda = \mathscr{L}(\mathbf{b}_1, \ldots, \mathbf{b}_n) \qquad L = \mathscr{L}(\mathbf{b}_1, \ldots, \mathbf{b}_k) \qquad N = \mathscr{L}(\mathbf{b}_1, \ldots, \mathbf{b}_{k+1})$$





After D-SVP-reduction:

$$\|\mathbf{b}_{k+1}^\star\|^{-1} = \lambda_1(N^\times).$$

Slide-inspired Reduction: Analysis

How does each Primal/Dual step change $\mathrm{vol}(L)$?

After the Primal step

$$\mathrm{vol}(N) = \mathrm{vol}(L)\lambda_1(\Lambda/L)$$

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After the Dual step

$$\mathrm{vol}(N) = \mathrm{vol}(L')\lambda_1(N^\times)^{-1}$$

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After the Primal step

$$\mathrm{vol}(N) = \mathrm{vol}(L)\lambda_1(\Lambda/L)$$

Finally

$$\frac{\mathrm{vol}(L')}{\mathrm{vol}(L)} = \lambda_1(\Lambda/L)\lambda_1(N^\times)$$

After the Dual step

$$\mathrm{vol}(N) = \mathrm{vol}(L')\lambda_1(N^\times)^{-1}$$

Slide-inspired Reduction: Analysis

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After the Dual step

$$\mathrm{vol}(N) = \mathrm{vol}(L')\lambda_1(N^\times)^{-1}$$

- If $\lambda_1(\Lambda/L)\lambda_1(N^\times) < 1 - \frac{1}{\mathrm{poly}(n)}$, we win!
- For general lattices, we can only use Minkowski's theorem to bound $\lambda_1(\Lambda/L)$ and $\lambda_1(N^\times)$.

Roadmap

Ducas' idea: bounding λ_1 for projections of $\mathbb{Z}^n$

Lemma (From [Duc23])

Let L be a sublattice of $\mathbb{Z}^n$ of rank k and volume $\mathrm{vol}(L) > 1$ such that $\pi_{L^\perp}(\mathbb{Z}^n)$ is a lattice, then

$$\lambda_1(\pi_{L^\perp}(\mathbb{Z}^n)) \leq \sqrt{1 - \frac{1}{n}}.$$

- Gives much stronger bound on $\lambda_1(\Lambda/L)\lambda_1(N^\times)$ than Minkowski's theorem.
- $\mathrm{vol}(L)$ decreases by at least $(1 - \frac{1}{n})$ at each Primal/Dual step.

A more general result and how to use it

Lemma

Let L be a sublattice of $\mathbb{Z}^n$ of rank k such that $\lambda_1(L) > 1$ and $\pi_{L^\perp}(\mathbb{Z}^n)$ is a lattice, then

$$\lambda_1(\pi_{L^\perp}(\mathbb{Z}^n)) \leq \sqrt{1 - \frac{k}{n}}.$$

- In particular if $k = \frac{n}{2}$, then $\lambda_1(\pi_{L^\perp}(\mathbb{Z}^n)) \leq \frac{1}{\sqrt{2}}$.

Modified algorithm: relaxing the approximation factor

Input: A bad basis of a hypercubic Λ

Main loop:

I. Check for unit vectors in L

II. γ-SVP reduce Λ/L

III. Check for unit vectors in $(\Lambda^\times/N)^\times$

IV. γ-Dual-SVP reduce N

Each line only uses a $\gamma < \sqrt{2}$ approximation oracle in halved dimension. $\mathrm{vol}(L)$ decreases by at least:

$$\gamma^2 \lambda_1(\Lambda/L)\lambda_1(N^\times) \leq \gamma^2/2 = 1 - \varepsilon.$$

Does it matter? Perspectives

- The best (provable and heuristic) algorithms for $\mathbb{Z}$LIP run in $2^{n/2+o(n)}$.
- For large enough (constant) γ, dim $n/2$ γ-SVP runs in $2^{0.401n+o(n)}$.

Open problems:

- What is the *real* cost of solving $\sqrt{2}$-SVP?
- Can we break the $n/2$ barrier for $\mathbb{Z}$LIP?
- Is the "easiest lattice" really that hard?

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Observation: a similar algorithm works more generally

Using exact-SVP-oracles: at each step $\mathrm{vol}(L)$ is multiplied by $\lambda_1(\Lambda/L)\lambda_1(N^\times)$.

Quick Lemma

If $\lambda_1(L) > \lambda_1(\Lambda)$, then $\lambda_1(\Lambda/L) \leq \lambda_1(\Lambda)$.

Consequence: Testing $\lambda_1(L) > \lambda_1(\Lambda)$ with an SVP-oracle

$\implies$ at each step $\mathrm{vol}(L)$ is multiplied by at most $\lambda_1(\Lambda)\lambda_1(\Lambda^\times)$.

Surely no reasonable lattice family satisfies $\lambda_1(\Lambda)\lambda_1(\Lambda^\times) < 1-\varepsilon$??

The symplectic nature of NTRU

Lemma (rescaled NTRU is isodual)

If Λ is a NTRU lattice with modulus q over a ring $\mathbb{Z}[X]/(X^n \pm 1)$, then Λ and $q\Lambda^\times$ are isometric.

For such lattices, $\lambda_1(\Lambda)\lambda_1(\Lambda^\times) = \frac{\lambda_1(\Lambda)^2}{q}$.

So when is $\lambda_1(\Lambda)\lambda_1(\Lambda^\times) < 1-\varepsilon$??

Upper bound on $\lambda_1(\Lambda)\lambda_1(\Lambda^\times)$ for various NTRU parameters			
Lattice	$\lambda_1(\Lambda)\lambda_1(\Lambda^\times)$	$\frac{1}{2}\lambda_1(\Lambda)\lambda_1(\Lambda^\times)$	Approx factor
NIST-1 [CDH+20]	.2897	.1449	2.628
NIST-3 [CDH+20]	.3444	.1722	2.410
NIST-5 [CDH+20]	.2581	.1291	1.969
Falcon-512 [FHK+19]	1.341	.6706	1.251
Falcon-1024 [FHK+19]	1.342	.6708	1.250

Conclusion: Many NTRU instances are provably solvable with $n/2$ SVP oracles only.

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Comparing with Primal Attack Asymptotics

Asymptotically, how close are the best provable and heuristic estimates?

Lattice	Provable blocksize	Heuristic blocksize (GSA + 2016 est.)
Hypercubic	$n/2 + o(n)$	$n/2 - o(n)$
NTRU[1]	$n/2 + o(n)$	$4n/9 - o(n)$

This confirms that non-uniqueness of the shortest vector is not directly relevant to the optimal blocksize.

[1] Assuming $q = \Theta(n)$ and $\lambda_1(\Lambda) = \Theta(\sqrt{n})$.

Recap/Takeaway

Conclusions:

- Like $\mathbb{Z}^n$, NTRU's geometry makes it easier to provably reduce.
- We give an algorithm that uses dim $n/2$ SVP-oracles.
- Those oracles can be relaxed by a constant γ.
- We help reduce the gap between provable and heuristic results.

The End

Bonus questions:

- Which of NTRU and $\mathbb{Z}$LIP is easier?
- Can we exploit isoduality better?
- Can Primal/Dual reduction be made practical?

Check out the paper at:

`iacr.org/2024/601`.
(revision very soon)

Thank you
For listening! :-)

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