

# Fiat-Shamir with Aborts and Polytopes

Séminaire Cryptologie & Sécurité - Caen

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11th of June 2025



THALES



- Talk based on <https://eprint.iacr.org/2024/411.pdf>.

I. Intro: Fiat-Shamir and Rejection Sampling

II. The Polytope-based Framework

III. Choosing a Polytope  $\mathcal{H}$

IV. Sampling in  $\mathcal{H} \cap \mathbb{Z}^n$

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# Intro: New Standards in Quantum-Safe Crypto

- Shor's quantum algorithm threatens the RSA cryptosystem.
- This led to the rise of lattice crypto (1996 → today)!

**Federal Information Processing Standards Publication 204**

Published: August 13, 2024

Effective: August 13, 2024

**Announcing the  
Module-Lattice-Based Digital Signature Standard**

Figure: ML-DSA (Dilithium)

# How to hide a secret?

- If  $s \in \mathbb{Z}_q^\times$ :
  - ① Sample  $r \xleftarrow{\$} \mathbb{Z}_q^\times$ ;
  - ② Return  $r \cdot s$ .

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  - ② Return  $z = y + s$ ;



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- 1 Sample  $r \xleftarrow{\$} \mathbb{Z}_q^\times$ ;
- 2 Return  $r \cdot s$ .

- What if  $s \in [-1, 1]$ ?

- 1 Sample  $y \xleftarrow{\$} [-10, 10]$ ;
- 2 Return  $z = y + s$ ;
- 3 Restart if

$$z \in \bigcap_{x \in [-1, 1]} x + [-10, 10].$$

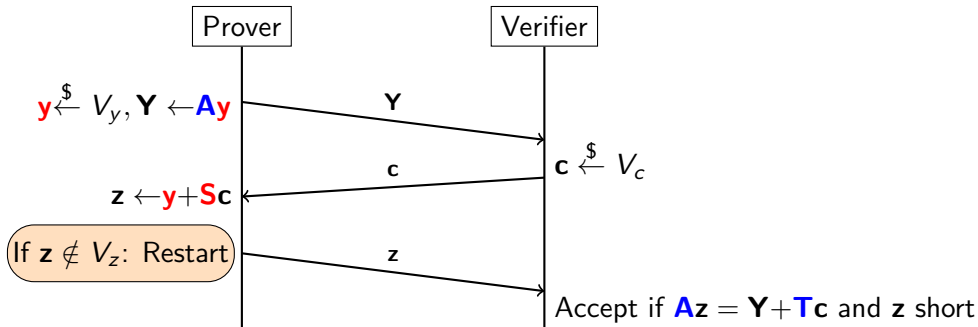
# SIS-based ID protocol

Secret parameters:

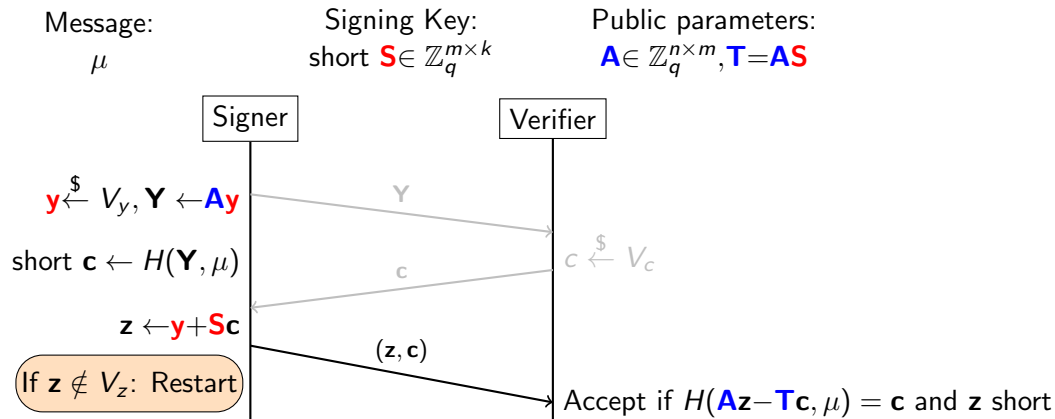
short  $\mathbf{S} \in \mathbb{Z}_q^{m \times k}$

Public parameters:

$\mathbf{A} \in \mathbb{Z}_q^{n \times m}, \mathbf{T} = \mathbf{AS}$

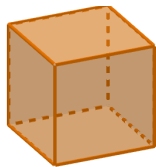


# Fiat-Shamir with Aborts

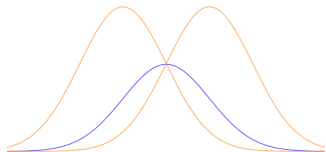


# Rejection sampling: a brief history of distributions

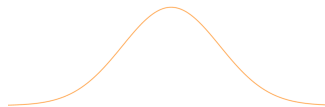
Idea: provably transform an imperfect distribution into a perfect distribution.



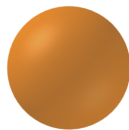
[Lyu09, DKL<sup>+</sup>21]



[DDLL13, CCD<sup>+</sup>23]



[Lyu12]



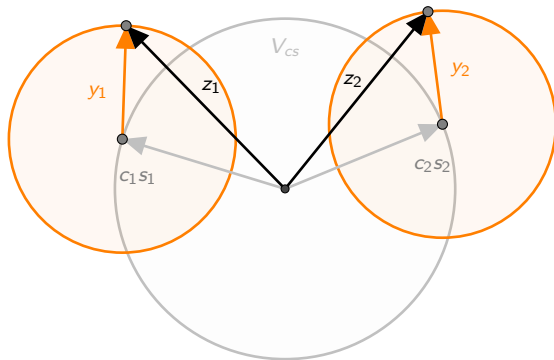
[CCD<sup>+</sup>23]

Our security relies on structured variants of SIS:  
MLWE, MSIS and SelfTargetMSIS.

The important metric for signature size and  $\text{Supp}(V_{cs})$  is the  $L_2$  metric.

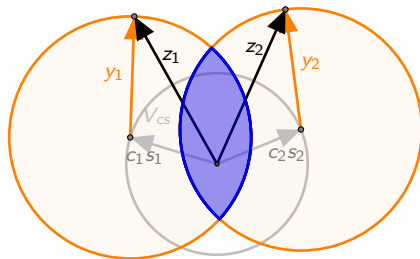
- We focus on the unimodal case (for now).
- We focus on uniform distributions.
- Notation: we identify distribution  $V_y$  and set  $\text{Supp}(V_y)$ .

## Rejection sampling: motivation



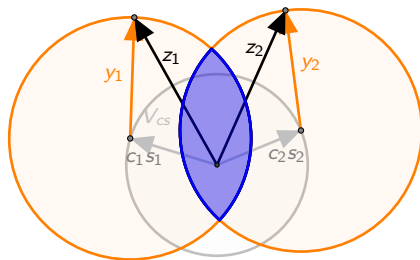
Knowing  $z$  should reveal **no information** on  $y$  and  $cs$ .

## Rejection sampling: motivation



**Witness-Indistinguishability:** each  $z$  in the **blue area** is equally likely to have been generated from any valid secret key.

# Rejection sampling: motivation



**Witness-Indistinguishability:** each  $z$  in the **blue area** is equally likely to have been generated from any valid secret key.

This must hold for **all** elements of  $V_{cs}$ .



# What do we want?

Assuming uniform distributions  $\mathbf{z}$  avoids information leakage if and only if:

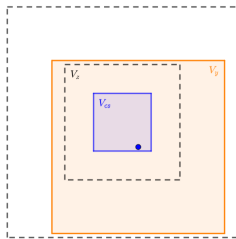
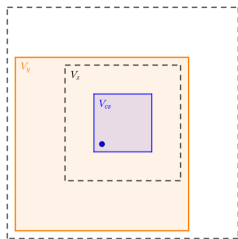
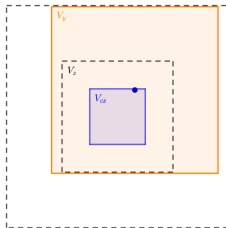
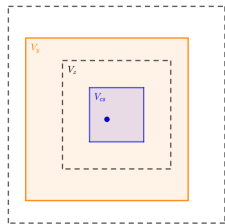
$$V_z \subseteq \bigcap_{\mathbf{x} \in V_{cs}} (V_y + \mathbf{x}).$$

Furthermore,  $V_z$  minimises the number of rejects if and only if:

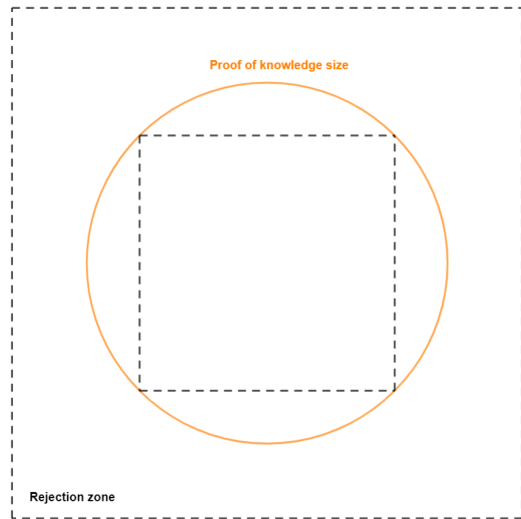
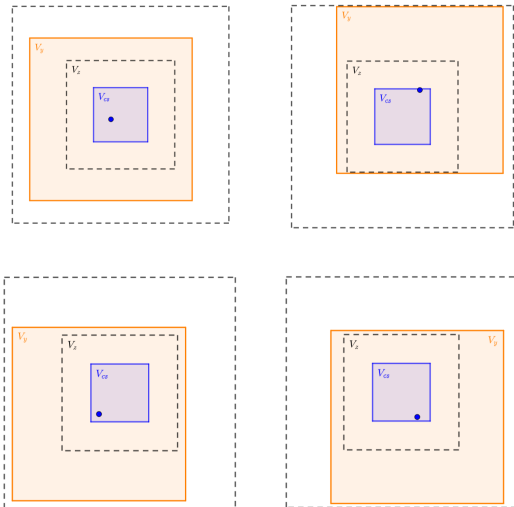
$$V_z = \bigcap_{\mathbf{x} \in V_{cs}} (V_y + \mathbf{x}).$$

- $\max_{\mathbf{z} \in V_z} \|\mathbf{z}\|_2$  conditions the signature size.
- Rejection rate depends on the tightness of the inclusion.

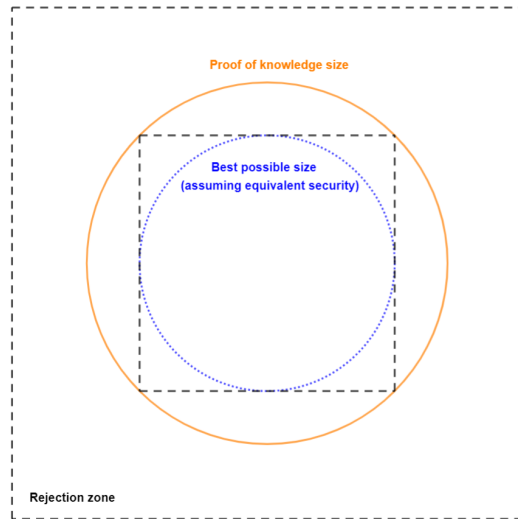
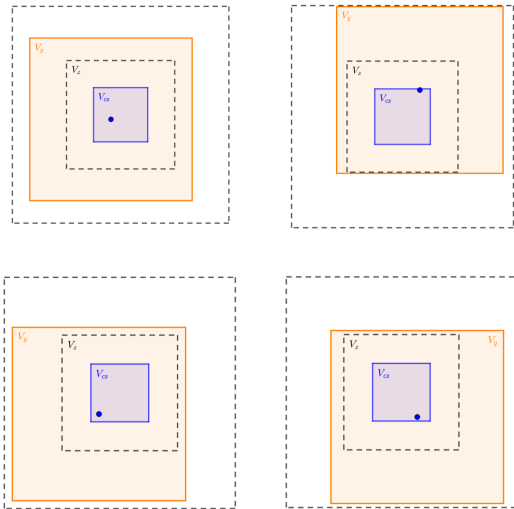
# Illustration: a Square



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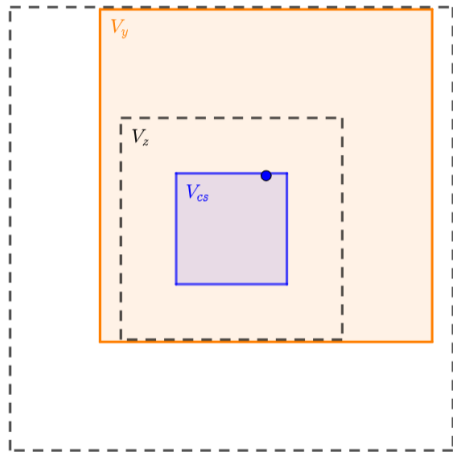
# Illustration: a Square



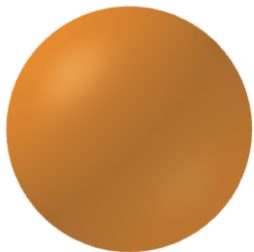
Probability of rejecting:

$$\frac{\text{Vol}(V_z)}{\text{Vol}(V_y)}.$$

- [DFPS22] observe that **Gaussian** distributions and uniform distributions in **Hyperballs** give optimal sizes.



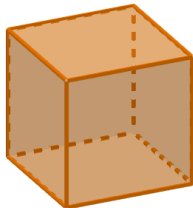
# Hyperballs: Pros and Cons



- Very small sizes (optimal according to [DFPS22]).
- Hard to mask against side channels.
- Hard to sample (Fixed point arithmetic).
- Only analysed in the continuous setting.
- Used in HAETAE [CCD<sup>+</sup>23].

# Hypercubes: Pros and Cons

- Larger sizes (in some sense hard to do worse).
- Easy to mask against side channels.
- Very simple sampler.
- Valid in the discrete setting.
- Used in DILITHIUM [DKL<sup>+</sup>21].



# Proposing a tradeoff: Objectives

**What we want:**

- Good proof sizes (better than DILITHIUM).
- A simple sampler (no FP arithmetic and no Gaussians).
- A valid analysis in the discrete setting.



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# Our solution: Polytopes

## Definition (Polytope)

A *polytope* is the convex hull of its vertices  $\mathcal{V}(\mathcal{P}) = \{\mathbf{x}_1, \dots, \mathbf{x}_v\} \in \mathbb{R}^n$ .



$$\mathcal{P}_{1,0}^n$$



$$\mathcal{P}_{1,(2,0,0)}^n$$



$$\mathcal{P}_{2,0}^n$$

# Polytope intersection: a useful tool

## Theorem ( $\mathcal{P}$ -ception: Intersection of polytopes)

Let  $\mathcal{P}$  be a symmetric inscriptible and circumscribable polytope. Let  $r, R \in \mathbb{R}_{>0}$  such that  $R > r$ . Then:

$$\bigcap_{\mathbf{c} \in \mathcal{P}_r} \mathcal{P}_{R,\mathbf{c}} = \bigcap_{\mathbf{c} \in \mathcal{V}(\mathcal{P}_r)} \mathcal{P}_{R,\mathbf{c}} = \bigcap_{\text{one } \mathbf{c}_i \text{ per facet of } \mathcal{P}_r} \mathcal{P}_{R,\mathbf{c}_i} = \mathcal{P}_{R-r}.$$

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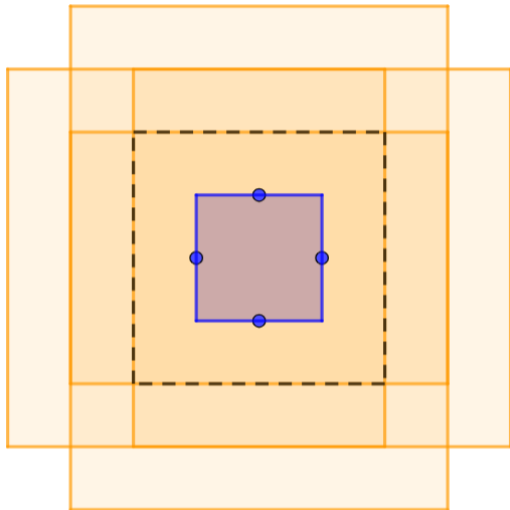
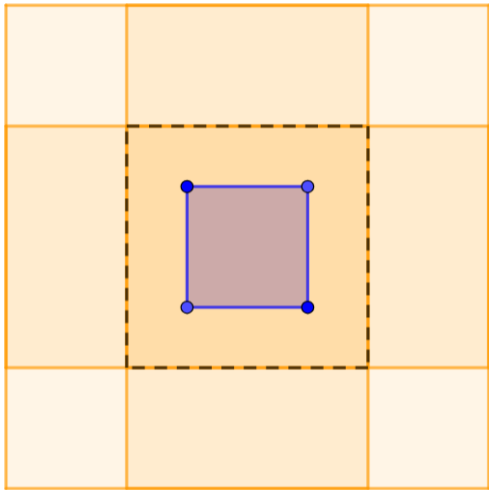
## Corollary (Discrete version)

If  $\mathcal{V}(\mathcal{P}_r) \subset \mathbb{Z}^n$ , then

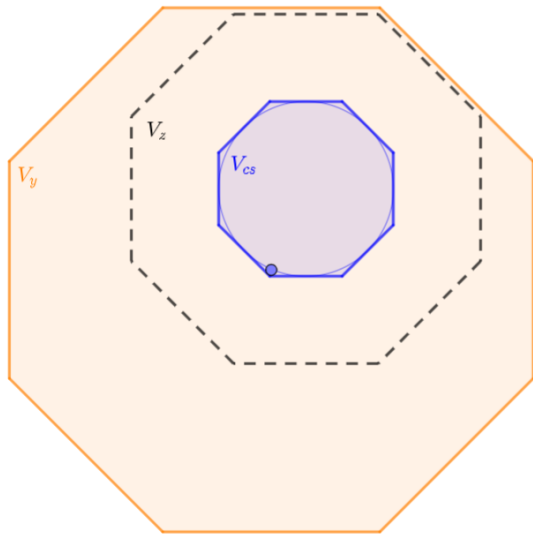
$$\bigcap_{\mathbf{c} \in \mathcal{P}_{r,\mathbb{Z}}} \mathcal{P}_{R,\mathbf{c}} = \bigcap_{\mathbf{c} \in \mathcal{V}(\mathcal{P}_r)} \mathcal{P}_{R,\mathbf{c},\mathbb{Z}} = \mathcal{P}_{R-r,\mathbb{Z}},$$

where  $\mathcal{P}_{\mathbb{Z}} = \mathcal{P} \cap \mathbb{Z}^n$ .

## $\mathcal{P}$ -ception: Illustration 1



## $\mathcal{P}$ -ception: Illustration 2



# Rejection Sampling with Polytopes: Continuous case

Let  $\mathcal{P}^n$  be a symmetric polytope whose vertices all lie on a sphere.

## Theorem (informal)

If  $V_y = \mathcal{P}_R^n$  and  $V_{cs} \subseteq \mathcal{P}_r^n$ , then:

$$\frac{\text{Vol } \mathcal{P}_{R-r}^n}{\text{Vol } \mathcal{P}_R^n} = \left( \frac{R-r}{R} \right)^n$$

determines the rejection rate.

In practical instantiations,  $r \ll R$ .

# Rejection Sampling with Polytopes: Discrete case

Let  $\mathcal{P}^n$  be a symmetric polytope, with **integral vertices** all on a sphere, then:

## Theorem (informal)

If  $V_y = \mathcal{P}_R^n \cap \mathbb{Z}^n$  and  $V_{cs} \subseteq \mathcal{P}_r^n \cap \mathbb{Z}^n$ , then:

$$\frac{|\mathcal{P}_{R-r,\mathbb{Z}}^n|}{|\mathcal{P}_{R,\mathbb{Z}}^n|} = \frac{\text{Vol } \mathcal{P}_{R-r}^n}{\text{Vol } \mathcal{P}_R^n} \cdot \frac{|\mathcal{P}_{R-r,\mathbb{Z}}^n|}{|\mathcal{P}_{R-r}^n|} \cdot \frac{\text{Vol } \mathcal{P}_R^n}{|\mathcal{P}_{R-r,\mathbb{Z}}^n|} = \left(\frac{R-r}{R}\right)^n \frac{1 + \varepsilon_R}{1 + \varepsilon_{R-r}}$$

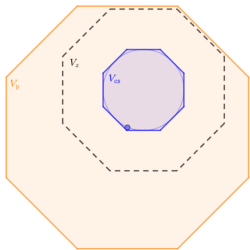
determines the rejection rate.

Computing  $\varepsilon_R$  and  $\varepsilon_{R-r}$  should be done only once, and requires:

- Volumes of integral polytopes.
  - Counting integral points in polytopes.
- } Efficient for well-chosen polytopes



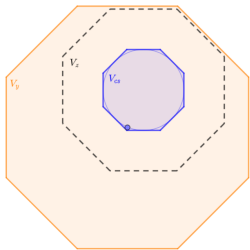
## Extra motivation: Optimality of rejection



Recall that we would like maximality of:

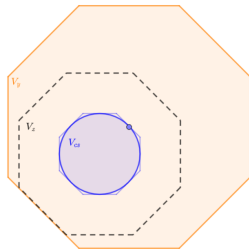
$$\bigcap_{\mathbf{x} \in V_{cs}} (V_y + \mathbf{x}).$$

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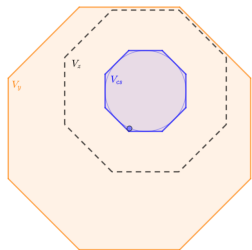


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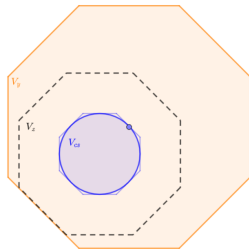


## Extra motivation: Optimality of rejection



Recall that we would like maximality of:

$$\bigcap_{\mathbf{x} \in V_{CS}} (V_y + \mathbf{x}).$$



If the support  $V_y$  is a polytope, and if  $\mathcal{P}$  is a symmetric polytope that admits an inscribed ball  $\mathcal{B}_2$  that is tangent to all of its facets, then we can interchangeably use  $\mathcal{P}$  or  $\mathcal{B}_2$  for the support of  $\mathbf{cs}$ .

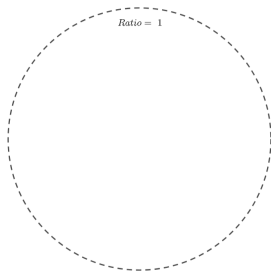
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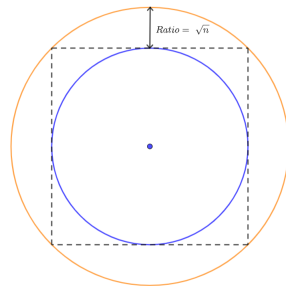
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# Polytope choice: Cutting a rare gem



## What we want for $\mathcal{P}$ :

- . Symmetric
- . Inscriptible
- . Circumscribable
- . Small ratio
- . Integral vertices
- . Efficiently samplable



## Interlude: High-dimensional balls



The Hypercube:

$$\mathcal{B}_{\infty}(R) = \{\mathbf{x} \in \mathbb{R}^n : \forall i, |x_i| \leq R\}.$$

- Norm:  $L_{\infty}$ .
- Volume:  $(2R)^n$ .
- Inradius:  $R$ .
- Circumradius:  $\sqrt{n}R$ .
- Mass concentrates: at the corners.
- $\int_{\mathbf{x} \in \mathcal{B}_{\infty}^n(R)} \|\mathbf{x}\|^2 d\mathbf{x} = nR^2/3$ .

## Interlude: High-dimensional balls

The Cross-polytope<sup>1</sup>:

$$\mathcal{B}_1(R) = \{\mathbf{x} \in \mathbb{R}^n : \sum |x_i| \leq R\}.$$

- Norm:  $L_1$ .
- Volume:  $\frac{(2R)^n}{n!}$ .
- Inradius:  $\frac{1}{\sqrt{n}} R$ .
- Circumradius:  $R$ .
- Mass concentrates: at the center.
- $\int_{\mathbf{x} \in \mathcal{B}_1^n(R)} \|\mathbf{x}\|^2 d\mathbf{x} \sim R^2/(2n)$ .

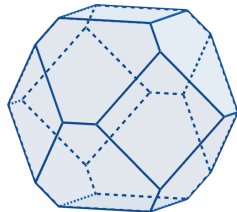
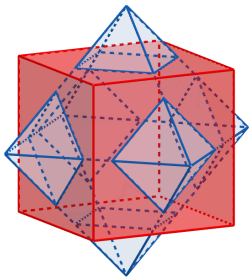
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<sup>1</sup>also called Hyperoctahedron, Orthoplex, or Cocube.



# The Polytope $\mathcal{H}$

$$\mathcal{H}_r^n = \mathcal{B}_\infty^n(r) \cap \mathcal{B}_1^n(r\sqrt{n})$$





# Some properties of $\mathcal{H}$

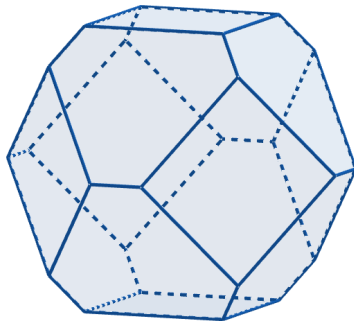
- Volume  $\approx \text{Vol}(\mathcal{B}_1^n(r\sqrt{n}))$  :

$$\frac{(2r\sqrt{n})^n}{n!} \sum_{i=0}^{\lfloor \sqrt{n} \rfloor} (-1)^i \binom{n}{i} \left(1 - \frac{i}{\sqrt{n}}\right)^{n+1}$$

- Inradius:  $r$  (by design).
- Circumradius:

$$r\sqrt{\lfloor \sqrt{n} \rfloor + (\sqrt{n} - \lfloor \sqrt{n} \rfloor)^2} \leq r\sqrt[4]{n}.$$

$\mathcal{H}$  is symmetric, and perfectly inscriptible and circumscribable.



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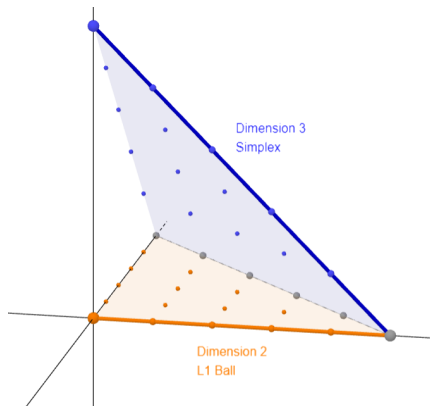
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# A useful projection

The following sets are isomorphic via a simple projection:

$$\mathcal{S}_{1,\mathbb{Z}^+}^{n+1}(r\sqrt{n}) = \{\mathbf{y} \in \mathbb{Z}_{\geq 0}^{n+1} : \|\mathbf{y}\|_1 = r\sqrt{n}\},$$

$$\mathcal{B}_{1,\mathbb{Z}^+}^n(r\sqrt{n}) = \{\mathbf{y} \in \mathbb{Z}_{\geq 0}^n : \|\mathbf{y}\|_1 \leq r\sqrt{n}\}.$$

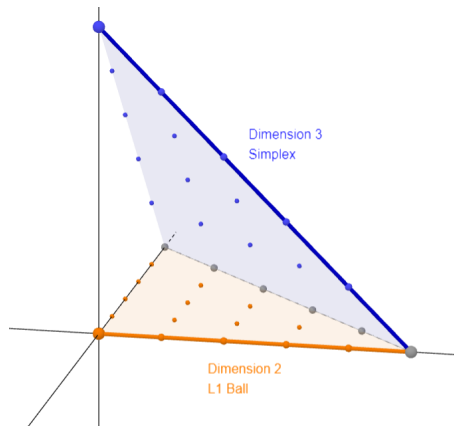


# Making the sampler Uniform and Isochronous

Mind the sides!

- Flip  $n$  coins for signs.
- Restart for each 0 coordinate, with probability  $1/2$ .

- Uniform: ✓
- Isochronous: ✓
- Expected restarts: small if  $n \ll r$ .

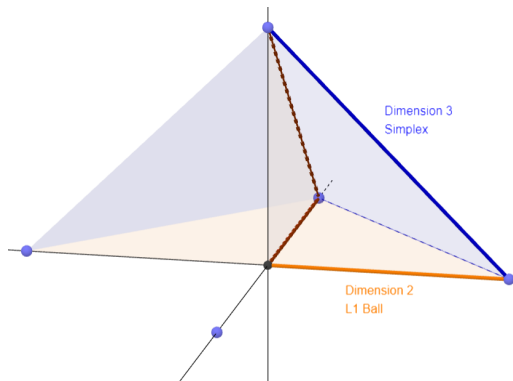


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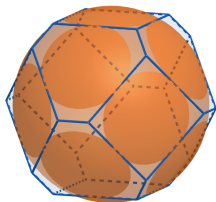
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- Restart for each 0 coordinate, with probability  $1/2$ .

- Uniform: ✓
- Isochronous: ✓
- Expected restarts: small if  $n \ll r$ .



We have simple sampling with quality  $n^{1/4}$

# Reject more for better performances



$$\mathcal{C}_{\theta,r}^n = \mathcal{H}_r^n \cap \mathcal{B}_2(\theta \cdot r)$$

where  $\theta \approx 1.5$

Key observation: for  $\theta > c$ ,

$$1 - \exp(-\sqrt{n}) < \frac{\text{Vol } \mathcal{C}_{\theta,r}^n}{\text{Vol } \mathcal{H}_r^n} < 1.$$

- Ratio  $n^{1/4} \rightarrow \theta$
- Trade-off between aborts and size.
- Warning: not a polytope anymore.

# A new Fiat-Shamir with Aborts signature scheme: PATRONUS





# Signature performances: Concrete parameters

- **Signature sizes:** (in bytes)

Security target (bits)	120	180	260
HAETAE	1,463	2,337	2,908
PATRONUS (this work)	2,070	2,575	3,721
DILITHIUM	2,420	3,293	4,595

- **Verification key sizes:** Similar to DILITHIUM ✓
- **Expected rejects:** Similar to HAETAE ✓
- **Sampler randomness:** at most 1.3 times that of DILITHIUM ✓
- **Sampler speed:** Slower than DILITHIUM - Faster than HAETAE

## What you should remember:

- We propose a new framework for rejection sampling in polytopes.
- This allows for rigorous analysis of perfect rejection in Fiat-Shamir.
- Our polytope  $\mathcal{H}$  uses  $L_1$  and  $L_\infty$  balls to approach an optimal  $L_2$  ball.
- It is easy to sample from  $\mathcal{H}_{\mathbb{Z}}$ .
- This leads to the signature scheme PATRONUS , an interesting tradeoff between DILITHIUM and HAETAE.

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V. Bonus: Open Questions and Perspectives

# Can we get a better polytope?

Theorem (From [Kas77])

*There exists a constant  $1 < c < 32$  such that for each  $n$ , there exists an orthogonal  $U \in \mathcal{O}_n(\mathbb{R})$  such that*

$$\mathcal{B}_2^n(1) \subseteq \mathcal{B}_1^n(\sqrt{n}) \cap U\mathcal{B}_1^n(\sqrt{n}) \subseteq \mathcal{B}_2^n(c).$$

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$\cap$



$=$



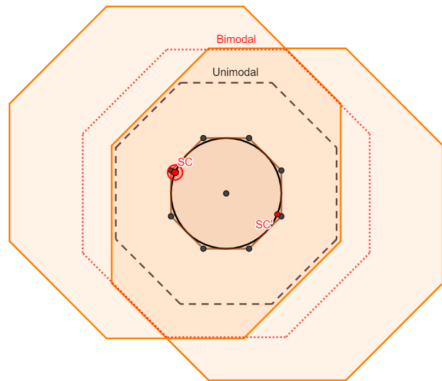
# The Bimodal situation

**Objective:** Use the trick by [DDLL13] for better sizes.

- We need to study

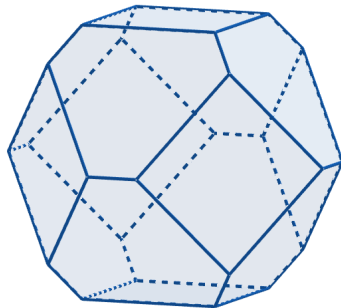
$$I = \bigcap_{\mathbf{sc} \in \mathcal{B}_2(r)} (\mathcal{P}_{R,\mathbf{sc}} \cup \mathcal{P}_{R,-\mathbf{sc}})$$

- No improvement in the Hypercube case.
- For  $\mathcal{H}$ , no obvious improvement after dim 4 as the largest  $\mathcal{H}$  in  $I$  is  $\mathcal{H}_{R-r}$ .
- For  $\mathcal{C}$ , less unlikely.



# The End

Thank you for listening!




If you have extra questions, feel free to contact Hugo ([hugo.beguinet@ens.fr](mailto:hugo.beguinet@ens.fr))

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