An inverse problem for isogeny volcanoes

LFANT Seminar

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Disclaimer

- Joint work with Francesco Campagna [1] and Fabien Pazuki [2]
- Talk based on https://arxiv.org/abs/2210.01086
- I am now a PhD student in lattice-based crypto

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Outline of the talk

- Introduction to isogeny graphs in the *easiest* setting
- The connected components: Volcano exploration
- Solving the inverse problem

Isogeny graphs: (brief) history and applications

Original work

- David Kohel's PhD thesis (1996)

A computational tool

- Computing endomorphism rings
- Computing modular/Hilbert class polynomials
- Point counting

In cryptography

- First proposal by Couveignes (1997)
- Post-Quantum attempts: SIDH, CSIDH, etc

Defining vertices: j-invariants

j-invariants

Let $E/\mathbb{F}_p : y^2 = x^3 + ax + b$ be an elliptic curve, the j-invariant of E is

$$j(E) = j(a, b) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

- p possible j-invariants, all are reached.
- Encompass classes of $\overline{\mathbb{F}}_p$-isomorphisms $(x, y) \mapsto (u^2 x, u^3 y)$.
- $j = 0$ and $j = 1728$ (in $\mathbb{F}_p$) are special.

Defining edges: isogenies

Isogenies

An isogeny is a non-constant homomorphism $\varphi : E \to E'$.
It is surjective and has finite kernel $C = \ker\varphi$.
The degree of φ is $\deg\varphi = \#C$.

- An isogeny φ is defined over $\mathbb{F}_p$ if $\ker\varphi$ is stable by Galois action.
- In this talk, isogenies are equivalent up to their kernel.
- Small degree isogenies are easy to compute.

Ordinary vs supersingular

Endomorphism ring

Let $E/\mathbb{F}_p$ be an elliptic curve, and k a field. Then the endomorphism ring $\mathrm{End}_k(E)$ is the ring of all k-rational isogenies from E to itself.

- When $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is an order in an imaginary quadratic field, E and $j(E)$ are called ordinary .
- The rest is supersingular.
- Over $\mathbb{F}_p$, we have $O(\sqrt{p})$ supersingular j-invariants.
- Every $\overline{\mathbb{F}}_p$-isogeny between ordinary curves with $j \neq 0, 1728$ has an equivalent $\mathbb{F}_p$-isogeny .

Quick reminder: imaginary quadratic orders

Orders are subrings of the ring of integers.

Maximal order

In $K = \mathbb{Q}(\sqrt{-D})$,
$\mathcal{O}_K = \mathbb{Z}[\sqrt{-D}]$ or $\mathbb{Z}\left[\frac{1+\sqrt{-D}}{2}\right]$.

Quadratic orders

Orders in $K = \mathbb{Q}(\sqrt{-D})$ are of the form $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ with $f \in \mathbb{Z}_{>0}$ (think lattices).

- We have a correspondence between negative integers $\equiv 0, 1 \mod 4$ and orders.
- $f = [\mathcal{O}_K : \mathcal{O}]$ is called the conductor of $\mathcal{O}$.
- $\mathrm{Disc}(\mathcal{O}) = f^2 \, \mathrm{Disc}(\mathcal{O}_K)$.
- We define $\mathrm{Cl}(\mathcal{O})$ as usual.
- Class number notation: $h(\mathcal{O}) = \# \, \mathrm{Cl}(\mathcal{O})$.

So what is the isogeny graph??

- $p > 3$ is a *large* prime.
- $\ell \neq p$ is a *small* prime.

Isogeny graph

The ordinary ℓ-isogeny graph $\mathcal{G}_\ell(\mathbb{F}_p)$ has set of vertices all ordinary j-invariants in $\mathbb{F}_p$ and edges all $\mathbb{F}_p$-rational ℓ-isogenies.

- Up to isomorphism of curves, up to equivalence of isogeny.
- $\mathcal{G}_\ell(\mathbb{F}_p)$ can be seen as undirected outside of $j = 0, 1728$.
- Possible self-loops, double edges and double self-loops.
- Roots of $\Phi_\ell(X, Y)$ with multiplicity.

Pictures!

Structure: Frobenius, trace and cordilleras

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The Frobenius equation

Let π be the Frobenius endomorphism associated to $E/\mathbb{F}_p$ where $j(E) \neq 0, 1728$ and $K = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$4p - t^2 = -f^2 \,\mathrm{Disc}(\mathcal{O}_K)$$

where $t = \mathrm{Tr}(\pi)$ and $f = [\mathcal{O}_K : \mathbb{Z}[\pi]]$.

- $j(E) \neq 0, 1728$ means $\mathrm{Disc}(K) < -4$ and $\#\,\mathrm{Aut}(E) = 2$: the equation in red has at most one solution $(t, f) \in \mathbb{N}^2$.
- In fact $t = p + 1 - \#E$: we have $|t| \leq \lfloor 2\sqrt{p} \rfloor$.
- Isogenies preserve $\#E$.
- Same Frobenius $\iff$ same $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ $\iff$ same trace up to sign $\iff$ isogenous up to equivalence.

Structure: Frobenius, trace and cordilleras

Cordillera

The *t-cordillera*[a] of $\mathcal{G}_\ell(\mathbb{F}_p)$ is the subgraph induced by the following set of vertices:

$$\mathcal{V}_t = \{j(E) : E/\mathbb{F}_p \text{ and } p + 1 - \#E(\mathbb{F}_p) = \pm t\}.$$

[a] Terminology credit: Miret, Sadornil, Tena, Tomàs and Valls (2007)

- 1 positive $t \iff$ 1 imaginary quadratic field (*).
- All $\mathcal{O} = \mathrm{End}_{\mathbb{F}_p}(E)$ for $E/\mathbb{F}_p$ such that $j(E) \in \mathcal{V}_t$ satisfy

$$\mathbb{Z}[\pi_t] \subseteq \mathcal{O} \subseteq \mathcal{O}_K.$$

- All *ordinary* traces live in $[\![1, \lfloor 2\sqrt{p} \rfloor]\!]$.
- There can be no edges between cordilleras (*).
- Over $\mathbb{F}_p$, no cordillera is empty (Waterhouse 1969).

Structure: Horizontal vs Vertical isogenies

Lemma

Let $\varphi : E_1 \to E_2$ be an ℓ-isogeny. Then

$$[\mathcal{O}_1 : \mathcal{O}_2] = \frac{1}{\ell}, 1, \text{ or } \ell.$$

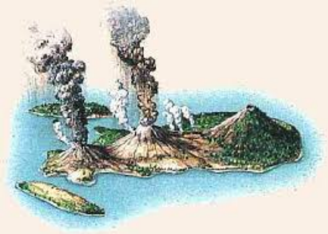
- φ increases $\mathcal{O}$: vertical ascending.
- φ decreases $\mathcal{O}$: vertical descending.
- φ leaves $\mathcal{O}$ unchanged: horizontal.

$$\mathbb{Z}[\pi] \subseteq \mathbb{Z} + m\ell^d \mathcal{O}_K \subset \mathbb{Z} + m\ell^{d-1}\mathcal{O}_K \subset \ldots \subset \mathbb{Z} + m\mathcal{O}_K \subseteq \mathcal{O}_K$$

Structure: Volcano Belts

Belts

We partition cordilleras into belts: subgraphs in which all orders have conductors of the form $m\ell^k$, where m is coprime to ℓ.



$$\mathbb{Z}[\pi] \subseteq \mathbb{Z} + m\ell^d \mathcal{O}_K \subset \mathbb{Z} + m\ell^{d-1}\mathcal{O}_K \subset \ldots \subset \mathbb{Z} + m\mathcal{O}_K \subseteq \mathcal{O}_K$$

In a given cordillera,

$$\{\text{belts}\} \longleftrightarrow \{\text{divisors of the conductor of } \mathbb{Z}[\pi] \text{ coprime to } \ell\}$$

Structure: Levels and ascending isogenies

Levels

A vertex of $\mathcal{G}_\ell(\mathbb{F}_p)$ with order $\mathbb{Z} + m\ell^k\mathcal{O}_K$ lies at level k if $(\ell, m) = 1$. If $\mathbb{Z}[\pi] = \mathbb{Z} + f\mathcal{O}_K$ then $d = v_\ell(f)$ is called the depth.

An ℓ-cordillera and its belts have a unique depth (*).

Lemma

Let $E/\mathbb{F}_p$ with $\mathrm{End}(E) = \mathbb{Z} + v\mathcal{O}_K$, where $\ell | v$. Then there exists a vertical ascending ℓ-isogeny from $j(E)$.

Structure: How many curves at a given level?

Lemma

Let $\mathcal{O}$ be an order of discriminant D in $K = \mathbb{Q}(\sqrt{t^2 - 4p})$ where $|t| \in [\![1, \lfloor 2\sqrt{p} \rfloor]\!]$. If $\mathbb{Z}[\pi] \subset \mathcal{O}$ Then the set $\mathrm{Ell}_{F_p}(\mathcal{O})$ of j-invariants with endomorphism ring $\mathcal{O}$ has cardinality $h(\mathcal{O}) = \# \mathrm{Cl}(\mathcal{O})$.

- These can be seen as roots mod p of the Hilbert class polynomial $H_D(X)$.
- Summing over all belts we can decompose p as a sum of class numbers.

Lemma

$$h(\mathcal{O}') = h(\mathcal{O}) \left(\ell - \left(\frac{\mathrm{Disc}(\mathcal{O})}{\ell} \right) \right) \quad \textit{if } [\mathcal{O}' : \mathcal{O}] = \ell$$

.

CM action and Horizontal isogenies

Lemma

- *If* $\varphi : E \to E'$ *is a horizontal* ℓ*-isogeny, there exists an integral invertible* $\mathcal{O}$*-ideal* $\mathfrak{L}$ *of norm* ℓ *such that* $E' \cong E/E[\mathfrak{L}]$.
- *Reciprocally, invertible ideals* $\mathfrak{L}$ *of norm* ℓ *give rise to* ℓ*-isogenies* $\varphi : E \to E/E[\mathfrak{L}]$.

- This is the degree ℓ part of the free and transitive group action of $\mathrm{Cl}(\mathcal{O})$ on $\mathrm{Ell}_{\mathbb{F}_p}(\mathcal{O})$.
- Now we only need to look at ideals!

Structure: Horizontal isogenies

Corollary

There are exactly $1 + \left(\frac{\mathrm{Disc}(\mathcal{O})}{\ell}\right)$ *horizontal edges from a vertex with endomorphism ring* $\mathcal{O}$*.*

- No horizontal isogenies outside of level 0!
- The level 0 only connected components are called craters.
- Otherwise the number only depends on the cordillera:

$$1 + \left(\frac{D(\mathcal{O}_K)}{\ell}\right) = \begin{cases} 0 & \text{if } \ell \text{ is inert in } K, \\ 1 & \text{if } \ell \text{ is ramified in } K, \\ 2 & \text{if } \ell \text{ splits in } K, \end{cases}$$

Structure: The crater

Structure: The crater

1. ℓ is inert in K
2. $\ell = \mathfrak{L}^2$, $\mathfrak{L}$ principal
3. $\ell = \mathfrak{L}\overline{\mathfrak{L}}$, $\mathfrak{L}$ principal
4. $\ell = \mathfrak{L}^2$, $\mathfrak{L}$ non principal
5. $\ell = \mathfrak{L}\overline{\mathfrak{L}}$, $[\mathfrak{L}]$ of order $n > 1$ in $\mathrm{Cl}(\mathcal{O})$

All craters in a given belt are the same, as cosets of the CM action.

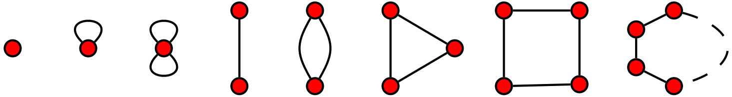


Figure: All possible craters.

Structure: The volcano

Structure: Degree of the vertices

Lemma

Let $j(E) \in \mathbb{F}_p$ be an ordinary j-invariant. Then the number of vertices from $j(E)$ in $\mathcal{G}_\ell(\mathbb{F}_p)$ is one of $0, 1, 2$ or $\ell + 1$.

Proof

- $\hat{\varphi} \circ \varphi = [\ell] \implies \ker \varphi \subset \ker[\ell]$
- $\ell + 1$ size ℓ subgroups of $E[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^2$
- Defined over $\mathbb{F}_p \implies$ invariant under $\mathrm{Gal}(\mathbb{F}_p(E[\ell])/\mathbb{F}_p)$
- Fixing ≥ 3 $\mathbb{F}_\ell$-lines of $(\mathbb{Z}/\ell\mathbb{Z})^2$ fixes everything.

Structure: Volcanoes

Theorem (Kohel)

Connected components (*) of $\mathcal{G}_\ell(\mathbb{F}_p)$ are ℓ-volcanoes [a]: a cycle (crater) with isomorphic trees (lava flows) at each of its vertices. All vertices have arity $\ell + 1$, except for the leaves of the trees.

[a] Terminology credit: Fouquet, Morain

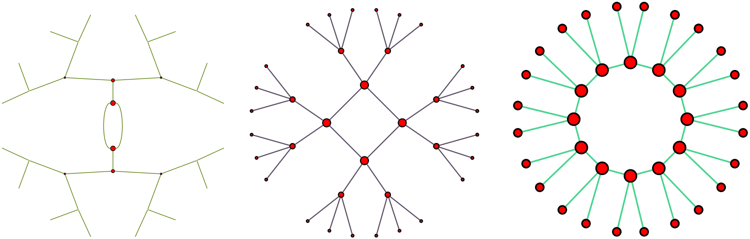


Figure: A 2-volcano and two 3-volcanoes

A zoo of possible connected components

Question: Suppose we are given an *abstract volcano* V[3]. Can we guarantee the existence of primes $p \neq \ell$ such that V is a connected component of $\mathcal{G}_\ell(\mathbb{F}_p)$?

[3] in the graph theoretic sense

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Full volcano: (V_0, ℓ, d)

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Crater only: $(V_0, \ell, 0)$

Full volcano: (V_0, ℓ, d)

Replace $\mathbb{F}_p$ with $\mathbb{F}_{p^r}$

[3] in the graph theoretic sense

A very useful trick: depth is not a problem

Lemma

If we can find an order $\mathcal{O}$ *of an imaginary quadratic field* K *with* $\ell \nmid \mathrm{Disc}(\mathcal{O}) < -4$, *and a prime (integral ideal)* $\mathfrak{L}$ *above the (rational) odd prime* ℓ, *such that* $\mathfrak{L}$ *would generate a crater* V_0, *then for any* $d \geq 0$, *the volcano* (V_0, ℓ, d) *exists in infinitely many isogeny graphs* $\mathcal{G}_\ell(\mathbb{F}_p)$.

- *would generate* can be well defined.
- If $\ell = 2$ the result only holds for $d > 0$.
- What this means: in practice, don't worry about p or d.

Depth is not a problem: sketch of proof

- We want $(t, f) \in \mathbb{N}^2$ and p such that

$$4p = t^2 - f^2 \operatorname{Disc}(\mathcal{O}),$$

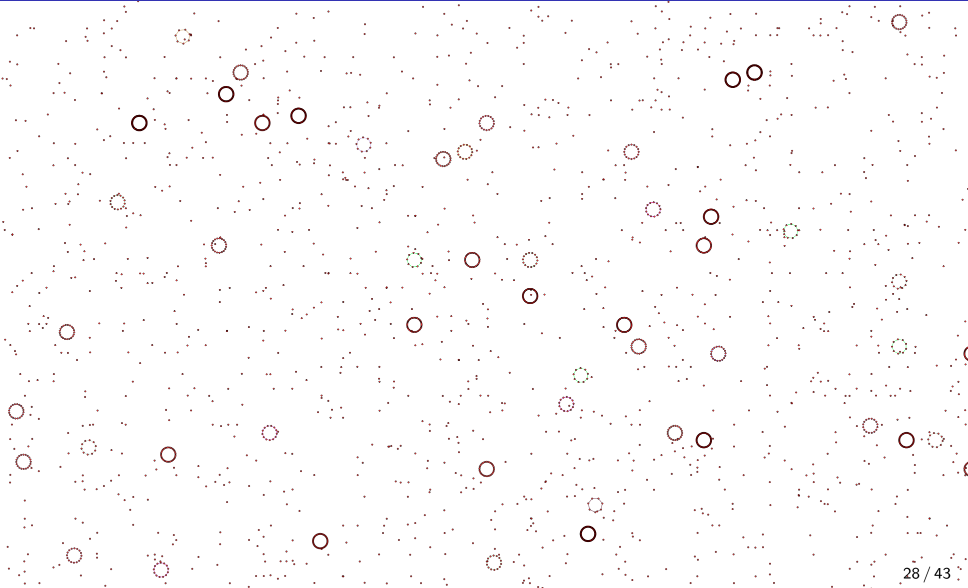
$t \neq 0$ and $v_\ell(f) = d$. This ensures $(V_0, \ell, d) \subset \mathcal{G}_\ell(\mathbb{F}_p)$.

- $p = x^2 + ny^2$ iff p splits completely in the ring class field of $\mathbb{Z}[\sqrt{-n}]$ (See Cox's eponymous book).
- Denote by H_k the ring class field of $\mathbb{Z}[\ell^k \sqrt{\operatorname{Disc}(\mathcal{O})}]$.

$$\begin{cases} H_d : & p = x^2 - \operatorname{Disc}(\mathcal{O})\ell^{2d}y^2 \\ H_{d+1} : & p = x^2 - \operatorname{Disc}(\mathcal{O})\ell^{2(d+1)}y^2 \end{cases}$$

- By Chebotarëv's theorem, there are infinitely many primes that split completely in H_d but not in H_{d+1}.

Intermission: funny behaviour?



Solving the weak inverse problem

Objective

Find infinitely many $p \neq \ell$ such that craters of size n are connected components in $\mathcal{G}_\ell(\mathbb{F}_p)$.

- Huge freedom on the choice of ℓ.
- Clear out all small craters by hand.
- Yamamoto (1970): we can construct an explicit imaginary quadratic field K such that $\mathrm{Disc}(\mathcal{O}_K) < -4$ and that has an element of order $n \geq 3$ in its class group $\mathrm{Cl}(\mathcal{O}_K)$.
- Cox (again!): the Dirichlet density of primes in a given quadratic imaginary class is strictly positive.
- Conclude with our previous Lemma.

Solving the inverse problem: easy craters

Objective

ℓ is now fixed. Find infinitely many $p \neq \ell$ such that volcanoes of shape (V_0, ℓ, d) are connected components in $\mathcal{G}_\ell(\mathbb{F}_p)$.

- Using our Lemma: forget about p and d , all we need is an imaginary quadratic field K in which ℓ has good behaviour.

Solving the inverse problem: easy craters

Objective

ℓ is now fixed. Find infinitely many $p \neq \ell$ such that volcanoes of shape (V_0, ℓ, d) are connected components in $\mathcal{G}_\ell(\mathbb{F}_p)$.

- Using our Lemma: forget about p and d , all we need is an imaginary quadratic field K in which ℓ has good behaviour.

Infinitely many K such that ℓ is inert (Dirichlet).

Figure: Crater type 1.

Solving the inverse problem: easy craters

Objective

ℓ is now fixed. Find infinitely many $p \neq \ell$ such that volcanoes of shape (V_0, ℓ, d) are connected components in $\mathcal{G}_\ell(\mathbb{F}_p)$.

- Using our Lemma: forget about p and d , all we need is an imaginary quadratic field K in which ℓ has good behaviour.

ℓ ramifies in a principal ideal of $\mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{-\ell})$. (*) for $\ell \leq 3$.

Figure: Crater type 2.

Solving the inverse problem: easy craters

Objective

ℓ is now fixed. Find infinitely many $p \neq \ell$ such that volcanoes of shape (V_0, ℓ, d) are connected components in $\mathcal{G}_\ell(\mathbb{F}_p)$.

- Using our Lemma: forget about p and d , all we need is an imaginary quadratic field K in which ℓ has good behaviour.

In $K = \mathbb{Q}(\sqrt{1-4\ell})$, $\alpha = \frac{1+\sqrt{1-4\ell}}{2}$ is integral of norm ℓ, who must split in $\mathcal{O}_K$ into two principal ideals.

Figure: Crater type 3.

Solving the inverse problem: easy craters

Objective

ℓ is now fixed. Find infinitely many $p \neq \ell$ such that volcanoes of shape (V_0, ℓ, d) are connected components in $\mathcal{G}_\ell(\mathbb{F}_p)$.

- Using our Lemma: forget about p and d , all we need is an imaginary quadratic field K in which ℓ has good behaviour.

Take $K = \mathbb{Q}(\sqrt{-\ell q})$ with a huge prime q. Then ℓ ramifies into a non-principal ideal, as its norm has to also be huge.

Figure: Crater type 4.

Solving the inverse problem: general craters

Objective

ℓ is now fixed. Find infinitely many $p \neq \ell$ such that volcanoes of shape (V_0, ℓ, d) are connected components in $\mathcal{G}_\ell(\mathbb{F}_p)$.

- Using our Lemma: forget about p and d , all we need is an imaginary quadratic field K in which ℓ has good behaviour.

Much harder! We want ℓ to split in two ideals whose class has prescribed order n in the ideal class group.

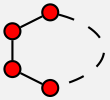


Figure: Crater type 5.

Solving the inverse problem: general craters

Theorem

The following properties hold.

1. *Let $n \neq 4$ be a positive integer and let $K = \mathbb{Q}(\sqrt{1 - 2^{n+2}})$. Then in $\mathcal{O}_K$ the prime 2 splits into two prime ideals whose corresponding classes in $\mathrm{Cl}(\mathcal{O}_K)$ have order n.*
2. *Let $K = \mathbb{Q}(\sqrt{-39})$. Then in $\mathcal{O}_K$ the prime 2 splits into two prime ideals whose corresponding classes in $\mathrm{Cl}(\mathcal{O}_K)$ have order 4.*
3. *Let $\ell \in \mathbb{Z}$ be an odd prime and let $n \in \mathbb{Z}_{>0}$. Define $K_1 := \mathbb{Q}(\sqrt{1 - \ell^n})$ and $K_2 := \mathbb{Q}(\sqrt{1 - 4\ell^n})$. Then either in $\mathcal{O}_{K_1}$ or in $\mathcal{O}_{K_2}$ the prime ℓ splits into two prime ideals whose corresponding classes in $\mathrm{Cl}(\mathcal{O}_{K_i})$ have order n.*

Solving the inverse problem: sketch of proof

- We work directly with diophantine equations.
- We use results from Nagell, Mahler and Pell.
- For example if $\ell = 2$, and $K = \mathbb{Q}(\sqrt{1-2^{n+2}})$ we write $\sqrt{1-2^{n+2}} = x\sqrt{-A}$ with A squarefree:

$$(\mathfrak{L}\overline{\mathfrak{L}})^n = 2^n = \frac{Ax^2+1}{4} = \frac{(1+x\sqrt{-A})}{2}\frac{(1-x\sqrt{-A})}{2}$$

- Now $\mathrm{ord}_{\mathrm{Cl}(\mathcal{O}_K)}(\mathfrak{L})|n$. Suppose it is $q < n$.
- If $q = 2$ expand and start cooking to get a contradiction except in one special case.
- If q is odd after clever manipulations we reach

$$U^2 - DV^2 = -A,$$

whose solutions are given by a theorem of Mahler. With a little more work we get a contradiction.

Solving the inverse problem: sketch of proof

- The case where ℓ is an odd prime is fun.
- Similar manipulations combined with an idea from Nagell yield the following:
- $K_1 = \mathbb{Q}(\sqrt{1-\ell^n})$ works when $\frac{\ell^{n/2}\pm 1}{2}$ is not a square.
- $K_2 = \mathbb{Q}(\sqrt{1-4\ell^n})$ works when $\ell^{n/2}$ is not the sum of two consecutive squares.
- Exercise: one condition has to be true!

Failing to solve the general inverse problem

Other fields

Almost everything we said on the structure of $\mathcal{G}_\ell(\mathbb{F}_p)$ transfers to $\mathcal{G}_\ell(\mathbb{F}_{p^r})$ for $r > 1$. Not true for the inverse problem!

Figure: The abstract volcano induced by $(2\text{-cycle}, 2, 1)$.

Proposition

The above volcano is an impossibility in any $\mathcal{G}_2(\mathbb{F}_{p^2})$.

Summary

In this talk:

- We defined the ordinary isogeny graph $\mathcal{G}_\ell(\mathbb{F}_p)$.
- We proved that its connected components look like volcanoes.
- We solved the inverse volcano problem over $\mathbb{F}_p$: every volcano exists in some $\mathcal{G}_\ell(\mathbb{F}_p)$.

Other directions:

- Inverse problem over $\mathbb{F}_{p^r}$.
- Given a volcano, which is the smallest field in which it lives?
- Better statistics on volcanoes.
- Faster algorithms to generate $\mathcal{G}_\ell(\mathbb{F}_p)$.
- A supersingular inverse problem?
- How many ℓ do you need to fully connect a cordillera?

Conclusion

Thank you![4]

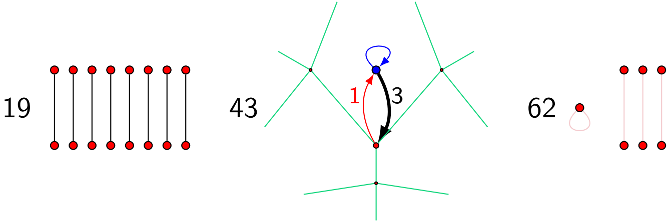


Figure: The 19/43/62-cordilleras in $\mathcal{G}_3(\mathbb{F}_{1009})$.

[4] If you want an illustration of any $\mathcal{G}_\ell(\mathbb{F}_p)$, feel free to send me an email!

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